

“Cathedral Builders”: Mathematics and the Sublime

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Unless the Lord builds the house,
the builders labour in vain.

Psalm 127

Σχᾶμα καὶ βᾶμα, ἀλλ’ οὐ σχᾶμα καὶ τριῶβλον.

Proclus, *In Euclidem*, 84.17¹

Mathematics, rightly viewed, possesses not only truth, but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show. The true spirit of delight, the exaltation, the sense of being more than man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry.

Bertrand Russell, *The Study of Mathematics*, 1902²

¹ “A figure and a stepping-stone, not a figure and three obols”. It is a Pythagorean proverb, according to Proclus. He provides us with an explanation of it: “By this they meant that we must cultivate that science of geometry which with each theorem lays the basis for a step upward and draws the soul to the higher world, instead of letting it descend among sensibles to satisfy the common needs of mortals and, in aiming at these, neglect to turn away hence”. Proclus, *A Commentary on the First Book of Euclid’s Elements*, transl. G.R. Morrow, Princeton, NJ 1992, p. 69.

² B. Russell, *Mysticism and Logic and Other Essays*, London 1917, p. 60.

He had no thought of beauties, but had already run beyond beauty [...], like a man who enters into the sanctuary and leaves behind the statues in the outer shrine; these become again the first things he looks at when he comes out of the sanctuary, after his contemplation within and intercourse there, not with a statue or image but with the Divine itself; they are secondary objects of contemplation.

Plotinus, *Ennead* VI.9.11³

Kann sich Gott nicht auch in der Mathematik offenbaren, wie in jeder anderen Wissenschaft?

Novalis⁴

Introduction⁵

Mathematical beauty is a widely discussed, though rather obscure topic. It can be traced back to Plato and early Pythagoreans but was powerfully revitalised around the turn of the 20th century. Bertrand Russell poeticised the “supreme beauty” of mathematics (see the third epigraph to this paper). Henri Poincaré announced the crucial role of aesthetic considerations—“the feeling of mathematical beauty, of the harmony of numbers and forms and of geometric elegance”—in the course of mathematical discovery.⁶ G.H. Hardy famously claimed:

The mathematician’s patterns, like the painter’s or the poet’s must be *beautiful*; the ideas like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics.⁷

³ Plotinus, with an English translation by A.H. Armstrong, In seven volumes, Vol. VII: *Enneads* VI. 6-9 (Loeb Classical Library), Cambridge, MA; London 1988, p. 343.

⁴ *Novalis Schriften*, Herausgegeben von Ludwig Tieck und Fr. Schlegel, Fünfte Auflage, Zweiter Teil, Berlin 1837, S. 148–149.

⁵ An early version of this paper was presented at the philosophy of mathematics workshop “Mathematical Aims beyond Justification” (Belgium, Brussels, 10-11 December 2015) organised by the Center for Logic and Philosophy of Science (CLWF) of the Vrije Universiteit Brussel.

⁶ H. Poincaré, *Mathematical Discovery* [1908], in: H. Poincaré, *Science and Method*, trans. F. Maitland, London 1914, p. 59.

⁷ G.H. Hardy, *A Mathematician’s Apology* [1940], with a foreword by C.P. Snow, Cambridge 1967, p. 85.

Hermann Weyl was praised by Freeman Dyson for his aesthetic sense and “profound faith in an ultimate harmony of Nature, in which the laws should inevitably express themselves in a mathematically beautiful form”.⁸ The list of similar examples could be continued almost without end.

Nevertheless, it is not so easy to tell what the word “beauty” means when applied to mathematics. According to Gian-Carlo Rota, “to be beautiful” stands for “to be enlightening”,⁹ while, say, Carlo Cellucci votes for “to provide understanding”.¹⁰ It is still a moot point. In this paper, I discuss mathematical beauty through the consideration of “sublimity”. “Sublime” is an adjective often used along with “beautiful” when talking of mathematics. For instance, Russell characterises mathematical beauty as “sublimely pure” (see the epigraph). Reviel Netz drops a general remark on the subject:

[M]ost mathematicians feel that there are aesthetic qualities to the mathematical pursuit itself. The states of mind accompanying the search for mathematical results are often felt as sublime; an aesthetic study seems warranted.¹¹

Netz specifies his use of “sublime” later on in the same paper: the genre of Greek mathematical texts, “as a whole, possesses beauty in its sublime impersonality”, that is, in its claim to possess absolute objectivity and truth.¹²

The “sublime” was listed alongside with “beauty” among the eighty factors analysed by Matthew Inglis and Andrew Aberdein in their experimental study of mathematicians’ perceptions of the qualities of mathematical proofs. Finally, it was included within the aesthetic dimension of the qualities with a rather high correlation coefficient of 0.52 to “beautiful”.¹³ This means that in many cases the same proof is regarded by mathematicians as both beautiful and sublime. Unfortunately, we do not know whether mathematicians surveyed in that experiment understood “sublime” in some specific aesthetic sense or according to general

⁸ F.J. Dyson, *Prof. Hermann Weyl* [an obituary], “Nature” March 10, 1956, Vol. 177, No. 4506, p. 458.

⁹ G.-C. Rota, *The Phenomenology of Mathematical Beauty*, “Synthese” 1997, Vol. 111, No. 2, pp. 181–182.

¹⁰ C. Cellucci, *Mathematical Beauty, Understanding, and Discovery*, “Foundations of Science” 2015, Vol. 20, No. 4, pp. 339–355.

¹¹ R. Netz, *The Aesthetics of Mathematics: A Study*, in: *Visualization, Explanation and Reasoning Styles in Mathematics*, eds. P. Mancosu, K.F. Jørgensen, S.A. Pedersen, Dordrecht 2005, p. 254.

¹² *Ibid.*, p. 261.

¹³ M. Inglis, A. Aberdein, *Beauty Is Not Simplicity: An Analysis of Mathematicians’ Proof Appraisals*, “Philosophia Mathematica” 2015, Vol. 23, No. 1, p. 101.

unspecified usage of this word in contemporary English. “Sublimely beautiful” seems to be a standard collocation, but “sublime” means here simply “of very high quality and causing great admiration”¹⁴ or “of very great excellence or beauty”¹⁵, that is, works as no more than an intensifier. The initial meaning of the word “sublime” in aesthetic theories of Edmund Burke and Immanuel Kant (which contrasted “the sublime” with “the beautiful”) is preserved only as a related subsense “producing an overwhelming sense of awe or other high emotion through being vast or grand”.¹⁶

The subsense just mentioned reminds us that the sublime is related to religious feelings and the aesthetics of the infinite. It should be noted that relations between science (mathematics included) and the initial meaning of the sublime are still alive and well in the works of contemporary Christian scholars, as the examples of Charles Taylor, Robert Gilbert, and Grigory Gutner show.¹⁷ Some religiously-neutral philosophers also adopt this position. For instance, Peter D. Suber popularised the idea of the sublimity of mathematics using emotionally charged language as follows:

I am profoundly grateful that understanding infinity does not deprive it of its majesty. If the infinite were only interesting because of the paradoxes it generates, and the absorbing academic issues raised by the need to resolve them, then it would not be studied any more than self-reference, a prolific but more pedestrian engine of paradox. But the infinite is also majestic, one might say infinitely majestic. An hour under a clear sky at night, looking up, gives some sense of this. The depth of space is a wild blue yonder, not a true, perceived infinity. But it inspires contemplation of the true infinite, and the slightest brush with that idea is breath-taking, invigorating, expanding, lifting, calming, but also agitating, alluring, but also distant and magnificently indifferent. One reason to study mathematics is that you can get these feelings in broad daylight or indoors. There are many ways to become precise about

¹⁴ *Oxford Advanced Learner's Dictionary of Current English*, 7th ed., ed. S. Wehmeier, Oxford 2005, p. 1529.

¹⁵ *Oxford Dictionary of English*, 3rd ed., ed. A. Stevenson, Oxford 2010, p. 1773.

¹⁶ *Ibid.*

¹⁷ See C. Taylor, *A Secular Age*, Cambridge, MA 2007, esp. pp. 322–351; R. Gilbert, *Science and the Truthfulness of Beauty: How the Personal Perspective Discovers Creation*, Abingdon, UK 2018, pp. 1–10, 79–80; G. Gutner, *The Origin and Motivation of Scientific Knowledge: A Treatise on Wonder*, Moscow 2018 (in Russian), pp. 10, 47, 54–57, 65–67.

these feelings, and many ways to praise and honor the infinite. I’d like to use Kant’s term: it is *sublime*.¹⁸

In this paper, I attempt to introduce *mathematical sublime* as a topic in its own right. To this end, I raise the issue of mathematical aims and recall the widely-known parable of the three cathedral builders (sections 2 and 3). The parable gives me a suitable metaphor to discern three attitudes to doing mathematics (as far as the ultimate goal of this human activity is concerned): pragmatic, aesthetic, and theological or religious. The parable also provides me with handy architectural metaphors which are used throughout this paper. Surely, such a choice of metaphors in no way turns my writing into the one on architecture or architectural history. Talking of “cathedral builders” I use scare quotes to emphasise the metaphorical use of the phrase (please, revisit the title of this paper). “Cathedral builders” refers to mathematicians who share in religious or, at least, quasi-religious attitude to doing mathematics. In the following sections 4 and 5, I introduce the term “numinous mathematics” to clarify the specific meaning of theological (religious) attitude to mathematics. I also connect it with the idea of God as a supreme mathematician and architect. Then, in section 6, I discuss the meaning of “the sublime” in general and define “mathematical sublime” in particular.

Sections from 7 to 9 are devoted to a critical exposition, evaluation and re-thinking of Kant’s theory of the sublime. I pay special attention to the peculiar interrelations of the sublime, the infinite, and mathematics. Distinguishing between the three distinct but connected types of the infinite—*aesthetic, potential, and actual*—helps to elucidate the unique role the sublime plays in connecting art, mathematics, and theology. My interpretation of Kant bridges his notion of *symbol* with that of the Romantic era: “the infinite in the finite”. It also outlines a covert similarity between mathematics and sacred art (see sections 10 and 11). The final sections are devoted to a conjectural treatment of mathematical beauty as mathematical sublimity. I do it based on the bipolar structure of symbol. The sublime feeling emerges when the finite is revealed as a manifestation of the infinite in mathematical objects, mathematical theorems or mathematical proofs. It is very often mixed up and confused with the feeling of mathematical beauty. I also give some examples to show that mathematical beauty (mathematical sublimity in disguise) may be recognised as a quasi-religious phenomenon or a sub-

¹⁸ P. Suber, *Infinite Reflections*, “St. John’s Review” 1998, Vol. XLIV, No. 2, pp. 1–34. Available at <https://dash.harvard.edu/handle/1/3715468>

stitute for religious feelings. Now a word of caution is in order: in this paper, I do not analyse mathematical practice as such, but rather people's attitudes towards the mathematical practice.

From external to ultimate internal goals

The topic of mathematical aims and goals is not so widely discussed as the kindred one on scientific aims and goals.¹⁹ Nevertheless, there is a wide range of specific aims pursued by mathematicians as mathematicians. Moreover, external goals differ from multilevel internal ones.²⁰ External goals are immediate objectives of a mathematician's efforts. For example, Timothy Gowers, writing on "the general goals of mathematical research", enumerates a variety of external goals. They are solving equations, classification, generalisation, proving, and so on.²¹ Among internal goals, one can find systematisation, unification, explanation and justification. These more general goals, as well as those described by Gowers, are also epistemic goals. On the next level, one can ask why we are looking for, say, justification, what kind of justification is appropriate for our purposes, and what these purposes are. Here, one turns to non-epistemic or practical goals. Finally, one can try to uncover *the ultimate internal non-epistemic goals* of the mathematical activity. It is sensible to stop just before we lose the specificity of mathematical activity and get into the field of general human goals. There is no doubt that mathematicians are no exception as far as human nature is concerned.²²

In the literature on management, it is widely accepted to tell between operational goals, tactical goals, strategic goals, and mission²³. There, we have a four-

¹⁹ K.C. Elliott, D.J. McKaughan, *Nonepistemic Values and the Multiple Goals of Science*, "Philosophy of Science" 2014, Vol. 81, No. 1, pp. 1–21; A. Potochnik, *The Diverse Aims of Science*, "Studies in History and Philosophy of Science Part A" 2015, Vol. 53, pp. 71–80.

²⁰ Cf. M. Tomasello et al., *Understanding and Sharing Intentions: The Origins of Cultural Cognition*, "Behavioral & Brain Sciences" 2005, Vol. 28, p. 676.

²¹ T. Gowers, *Introduction*, in: *Princeton Companion to Mathematics*, ed. T. Gowers, Princeton, NJ 2008, pp. 47–76.

²² Cf. R. Collins, S. Restivo, *Robber Barons and Politicians in Mathematics: A Conflict Model of Science*, "Canadian Journal of Sociology / Cahiers canadiens de sociologie" 1983, Vol. 8, No. 2, pp. 199–227.

²³ A. McMillan, D. Hansler, *Mission and Vision Statements*, in: *Encyclopedia of Management*, 5th ed., ed. M.M. Helms, Detroit, MI 2006, pp. 556–557.

level hierarchy of goals. In contrast to successful business organisations, mathematical communities rarely have explicit mission and vision statements. The mission and vision often remain half-conscious or even completely unconscious, being passed to the next generation through social mechanisms of the community reproduction in the long and complicated educational process. Among the oldest examples of such a mission statement for mathematical activity is the Pythagorean byword quoted above as one of the epigraphs to this paper.

The “cathedral builders” parable

The same external goal may be motivated by diverse internal goals. One may recall an old parable, the parable of the three stonemasons working at the construction of a cathedral.²⁴ In my country, the story is usually associated with the builders of Chartres Cathedral in the 13th century. Their external goal was the same: to cut stones, giving them the shape required. But being asked what they were doing, they gave different answers. The first one said: “I am making a living”. The second one: “I am doing the best job of stonemasonry in the entire county”. The third one: “I am building a cathedral!” Their internal goals turned out to be quite diverse. In my view, this parable can be successfully applied to mathematicians and their activity.

The parable is by no means superficial; the archetypes of temple and temple builders have deep roots in European and Jewish mysticism.²⁵ I would like to interpret the parable as representing three attitudes to doing mathematics: *pragmatic*, *aesthetic* and *theological*. According to the first attitude, mathematics is something very useful in science and everyday life, no wonder it helps mathematicians to earn a living. According to the second one, a pure mathematician inhabits a sort of ivory tower and practices a kind of art for art’s sake. These two are well known. For instance, Oswald Veblen spoke on the dual character of mathematics as the American Mathematical Society president in 1924:

On the one hand, mathematics is one of the essential emanations of the human spirit,—a thing to be valued in and for itself, like art or poetry. [...] On

²⁴ P.F. Drucker, *The Practice of Management*, New York, NY 1993, p. 122.

²⁵ Cf. M.K. Schuchard, *Restoring the Temple of Vision: Cabalistic Freemasonry and Stuart Culture*, Leiden 2002.

the other hand, mathematics is the handmaiden and helper of the other sciences, both in their most abstract generalizations and in their most concrete applications to industry.²⁶

Aesthetic attitude tends to identify mathematics with *pure* mathematics while pragmatic attitude is more interested in *applications*. Aesthetic and pragmatic attitudes to mathematics are widely recognised; the theological one is less known.²⁷ Within a peculiar perspective of the parable, only the third stonemason should be called a *true* cathedral builder. By this, I do not mean to say that the other two attitudes should be less respected. I just dare interpret the metaphor of “building a cathedral” as theological attitude to doing mathematics according to another famous metaphor describing a cathedral as “theology in stone”.²⁸ Hereafter, the name of “cathedral builders” will be reserved for those and only those mathematicians who take a theological attitude towards doing their work. Next, some clarification of what the latter attitude actually means should be made.

Numinous mathematics

Treating *theological* (or more broadly, *religious*) attitude as a non-reducible one, I am in the track of Rudolf Otto. Otto argued that religious experience had a non-reducible core for which he coined the term “*numinous*”. He distinguished numinous from aesthetic categories but closely associated the former with one of the latter, *the sublime*: “the sublime’ [...] is an authentic ‘scheme’ of ‘the holy’”.²⁹ According to the Oxford Dictionary, “*numinous*” means “having a strong religious or spiritual quality; indicating or suggesting the presence of a divinity”.³⁰ I am going to use it here in this very sense.

The adjective “numinous” is sometimes applied to mathematics in the context of Pythagorean and Platonic tradition. For instance, describing Plato’s view,

²⁶ R.G.D. Richardson, *The First Josiah Willard Gibbs Lecture*, “Bulletin of the AMS” 1924, Vol. 30, No. 7, p. 289.

²⁷ Cf. S. Krajewski, *Theological Metaphors in Mathematics*, “Studies in Logic, Grammar and Rhetoric” 2016, Vol. 44(57): *Theology in Mathematics?*, eds. S. Krajewski, K. Trzęsicki, pp. 13–30.

²⁸ Cf. R. Kieckhefer, *Theology in Stone: Church Architecture from Byzantium to Berkeley*, New York, NY 2004.

²⁹ R. Otto, *The Idea of the Holy*, trans. J.W. Harvey, 2nd ed., New York, NY 1950, p. 47.

³⁰ *Oxford Dictionary of English*, ed. A. Stevenson, 3rd ed., Oxford 2010, p. 1219.

Richard Tarnas says that mathematical objects “are numinous and transcendent entities, existing independently of both the phenomena they order and the human mind that perceives them”.³¹ Marsha Keith Schuchard is also speaking of “numinous mathematics” in the context of medieval Jewish tradition that was inherited in the European Middle Ages:

It is perhaps one of the strangest ironies of history that this originally Jewish yearning for transmundane and numinous mathematics would find its greatest architectural expression in the towering Gothic cathedrals built by Christian stonemasons.³²

One can find a striking appeal for the recognition of numinous mathematics in Novalis’s “Mathematische Fragmente” (1799/1800):

The highest life is mathematics.—
There can be supremely ranked mathematicians who cannot calculate.—
One could be a great calculator without having an inkling of mathematics.—
The true mathematician is an enthusiast *per se*. Without enthusiasm, there is no mathematics.—
The life of the Gods is mathematics.—
All divine messengers must be mathematicians.—
Pure mathematics is religion.—
One only advances to mathematics through a theophany.—
Mathematicians alone are fortunate. The mathematician knows all. He could know it, even if he did not already.
All activity ceases when knowledge enters. The state of knowledge is eudaimonia, the blessed peace of contemplation—heavenly quietism.—
True mathematics is at home in the Orient. In Europe, it has degenerated into a purely technical science.—
Whoever does not take hold of a mathematical book with devotion, and read it as the word of God, fails to understand it.³³—

³¹ R. Tarnas, *The Passion of the Western Mind: Understanding the Ideas that Have Shaped Our World View*, New York, NY 1991, p. 11.

³² M.K. Schuchard, *Op. cit.*, p. 24.

³³ Novalis, *Op. cit.*, p. 147–148. English translation by David. W. Wood is available at: https://www.academia.edu/15762280/Novalis_Mathematical_Fragments

Shall we take Novalis's *hymns to mathematics* ("die Hymnen auf die Mathematik" as Wilhelm Dilthey named them) seriously? Is there some sense in his project aimed at "a fusion of mathematics and religion"³⁴ or rather recognition of their initial intimate connection?

God as an architect and mathematician

Novalis's romantic enthusiasm about mathematics has a profound historical background. The idea to view mathematics as "numinous" (that is "transmundane" and "transcendent") is closely associated with the idea of *God as a mathematician*. Plato's *Timaeus* can be read in this way. Its God-demiurge is the Father and the Maker, the Craftsman and the Architect (ὁ πατήρ, ποιητής, τεκταινόμενος, δημιουργός, 28c–29a), but also someone highly skilled in mathematics. The rational organisation was imposed on the world through numbers and perfect geometrical forms (53b)³⁵. The world soul, ordained to embed the reason in the body of the world, was endowed to meet its purpose with a perfect rhythm governed by numbers and their relations. In the next step, this rhythm was realised in the circular movement of the heavenly bodies and, through them, in the whole world. The four elements are organised with the help of the four regular solids while the fifth regular body (dodecahedron) is used for the world as a whole (55c) on par with the sphere. Keeping in mind the story of Platonic solids from *Timaeus*, Plutarch famously summed up: "God always geometrizes (ἀεὶ γεωμετρεῖν τὸν θεόν)".³⁶

This Platonic motive is also apparently recognisable in some of late sacred texts of Judaism (perhaps finally composed in Hellenistic Alexandria) which were incorporated in the Christian Old Testament. God is claimed there as the one, who "has arranged all things by measure and number and weight" (*Wisdom 11:20*). This motive was recognised in the Middle Ages even in *Proverbs* (8:27) where the personified Wisdom says of God the Creator, according to the King James Bible: "When he prepared the heavens, I was there: when he set a compass

³⁴ M. Dyck, *Novalis and Mathematics: A Study of Friedrich von Hardenberg's Fragments on Mathematics and Its Relation to Magic, Music, Religion, Philosophy, Language, and Literature*, Chapel Hill, NC 1960, pp. 80–81.

³⁵ Plato, *Complete Works*, ed. J.M. Cooper, Indianapolis, IN 1997, p. 1256.

³⁶ *Questiones convivales*, 8.2, 718b–720c.

upon the face of the depth”. The “compass” stays here for the Latin “gyrus” (in the Vulgate) and Hebrew “khug”, i.e., “circle”.³⁷ This image of God the Architect was already pictured in detail by Philo of Alexandria, who tried to merge Moses with Plato and Pythagoreans: to build a great city (this world) God first needs to have a detailed architectural project (an ideal plan or model of this world) in his mind³⁸.

The mathematical interpretation of the divine compass was also famously played up by William Blake. He gave a pair of compasses in the hands of both God the Creator and the greatest mathematician of the epoch, Isaac Newton, who succeeded in uncovering the guiding lines of God’s mathematical plan of the universe.³⁹ The recognition of the numinous in the mathematical enterprise is less obvious in the 19th and 20th centuries, but closer investigation shows it is still with us, though often in disguise.⁴⁰

Mathematical sublime

Speaking of the indirect means by which the numinous is expressed in art, Rudolf Otto wrote: “In the arts nearly everywhere the most effective means of representing the numinous is ‘the sublime’. This is especially true of architecture, in which it would appear to have first been realized”.⁴¹ In my opinion, this natural shift from the numinous to the sublime works not only with art but with mathematics as well. The “*numinous mathematics*” refers to the experience that we interpret as an anticipation of a meeting with the Divine through doing mathematics. (Please, revisit the two epigraphs to this paper taken from Bertrand Russell and Plotinus). It satisfies our need for self-transcendence (in the language of Abra-

³⁷ For a discussion of the development, variation and meaning of this theme in the medieval miniatures see J.B. Friedman, *The Architect’s Compass in Creation Miniatures of the Later Middle Ages*, “Traditio” 1974, Vol. 30, pp. 419–429.

³⁸ *On the Creation*, IV, 17–19. *The Works of Philo Judaeus*, transl. C.D. Yonge, Vol. I, London 1890, pp. 4–5. Cf. J.B. Friedman, *Op. cit.*, p. 425.

³⁹ See *The Ancient of Days* (the frontispiece of *Europe a Prophecy*, copy D, 1794) and *Newton* (1795). Both are available at the *William Blake Archive*, <http://www.blakearchive.org/>

⁴⁰ V. Shaposhnikov, *Theological Underpinnings of the Modern Philosophy of Mathematics*, Parts I and II, “Studies in Logic, Grammar and Rhetoric” 2016, Vol. 44(57): *Theology in Mathematics?*, eds. S. Krajewski, K. Trzęsicki, pp. 31–54, 147–168.

⁴¹ R. Otto, *Op. cit.*, p. 65.

ham Maslow). The “*mathematical sublime*” refers to the same feelings but being transferred from the religious to the aesthetic sphere. “The sublime” designates “an *experience*, that of *transcendence*, which has its origins in religious belief and practice”; even nowadays, it remains, among other things, “an experience with mystical-religious resonances”.⁴² The sublime forms a religious stratum within aesthetic values.

By the way, what is “the sublime”? To answer this question is by no means easy. *Prima facie* one calls something “sublime” if it provokes a special feeling, being a strange amalgam of terror and admiration, for which English has a word: “awe”. According to some present-day dictionaries, the sublime is “a quality of awesome grandeur in art or nature, which some 18th-century writers distinguished from the merely beautiful”.⁴³ “An idea associated with religious awe, vastness, natural magnificence, and strong emotion, which fascinated 18th-century literary critics and aestheticians. Its development marks the movement away from the clarity of neo-classicism towards Romanticism, with its emphasis on feeling and imagination; [...]”.⁴⁴ To get a more detailed answer, one needs to choose among the variety of fully-fledged theories of the sublime.

The sublime and the different types of the infinite

The most influential and well-known theory of the sublime was put forward by Kant. What is more, Kant decisively connected the sublime with *the infinite*: “Nature is [...] sublime in those of its appearances the intuition of which brings with them the idea of its infinity”.⁴⁵ The main thesis behind Kant’s interpretation of the sublime is *the conflict* between abilities of the human imagination and the intellect when they deal with the infinite or at least the very big. The problem can be easily demonstrated in a famous example that can be traced back to Descartes’s *Sixth Meditation*. We can easily tell the difference between, say, the triangle and the square both by the imagination and by the intellect. But if we take a chiliagon,

⁴² R. Doran, *The Theory of the Sublime from Longinus to Kant*, Cambridge 2015, p. 1.

⁴³ C. Baldick, *The Oxford Dictionary of Literary Terms*, 3rd ed., Oxford 2008, p. 321.

⁴⁴ *The Concise Oxford Companion to English Literature*, eds. M. Drabble, J. Stringer, D. Hahn, 3rd ed., Oxford 2007.

⁴⁵ I. Kant, *Critique of the Power of Judgment*, ed. P. Guyer, transl. P. Guyer, E. Matthews (The Cambridge Edition of the Works of Immanuel Kant), New York, NY 2000, p. 138.

which is a polygon with one thousand sides, and, say, myriagon (a polygon with ten thousand sides) the imagination fails to distinguish between them while the intellect stays sharp. So the chiliagon turns out to be sublime for us while the triangle is not!

The first attempt to approach the conflict in question seems to be made by Aristotle, when he decisively divorced the infinity with physics to engage it exclusively with mathematics: there is no such thing as (actually) infinite physical body; nevertheless, mathematicians may enjoy the (potential) infinity of numbers and magnitudes in their discourse for “one might think that one of us is bigger than he is and magnify him *ad infinitum*” (*Physics*, 208a).⁴⁶ Aristotle distinguishes physical existence (in reality) from mathematical existence (or rather *subsistence*, taking the last word after Alexius Meinong for meaningful existence just in thought).

Kant made the next step, trisecting thought into senses plus imagination, the understanding (die Verstand), and reason (die Vernunft). These three abilities deal with infinity, though each in its own way. *Actual infinity* is an idea of reason restricted to metaphysics and theology.⁴⁷ *Potential infinity* is a concept of understanding restricted to mathematics. Besides, let me introduce *sensible or aesthetic infinity*⁴⁸ restricted to arts and sciences. This last type of infinity is encountered when, as Kant puts it, “the imagination reaches its maximum and, in the effort

⁴⁶ Aristotle, *Complete Works*, ed. J. Barnes, Princeton, NJ 1984, Vol. 1, p. 354.

⁴⁷ The Greek account of the infinite included both *negative* and *positive* interpretations. The negative one, which dominated, understood ἀπειρον as “unlimited” or even “indefinite” and “indeterminate”, i.e., as a sort of *imperfection*. The positive one is marked by Anaximander’s use of the word, which seems to imply some sort of *perfection*. The positive interpretation can be found cursorily mentioned in Aristotle (*Physics*, III, 4): “if coming to be and passing away do not give out, it is only because that from which things come to be is infinite” (203b18–20). Here, the source of being is infinite in the positive sense of *inexhaustibility* of its creative power. It was Plotinus (VI, 9, 6, 1–13) who overtly contrasted the two meanings of being infinite: the negative one in mathematics and the positive one as being applied to the Good. On the next step, the Divine Infinity, taken in the positive sense of the word, was established in Christian theology by Cappadocian fathers, especially Gregory of Nyssa (*Contra Eunomium*, 1.169; 1.367), to become an integral part of the orthodoxy. Thus, actual infinity, rejected by Aristotle for the realms of physics and mathematics, finally made its home in the realms of metaphysics and theology. Cf. L. Sweeney, *Divine Infinity in Greek and Medieval Thought*, New York, NY 1992.

⁴⁸ Cf. the use of “sensible infinity” and “aesthetic’ infinity” in J. Rogozinski, *The Gift of the World*, in: *Of the Sublime: Presence in Question, Essays by J.-F. Courtine et al.*, Albany, NY 1993, pp. 149–150.

to extend it, sinks back into itself”.⁴⁹ Though Kant scrupulously distinguishes between sensibility, the understanding, and reason, they work together in accord. Hence, three types of infinity are also closely connected.

HUMAN ABILITIES	THE INFINITE	HUMAN ACTIVITIES
senses + imagination	aesthetic infinity (finite, but very big)	arts and sciences
the understanding	potential infinity	mathematics
reason	actual infinity	metaphysics and theology

The central conflict, according to Kant, is between the regulative ideas of reason asking for a final synthetic totality (the absolute whole of the external world of nature or the internal world of our self) and the possibilities of senses and imagination giving us no more than a finite series of images. It means that actual infinity is no more than a regulative idea of pure reason, while sensible infinity is just something very big, so big that our imagination fails to present anything bigger. That is, in *sensu stricto*, the only true infinity is the potential infinity of mathematics, but “the *feeling* of the infinite”,⁵⁰ the *sublime* feeling, we get only when all three of our intellectual abilities work in accord or, rather, clash together.

The sublime and mathematics

Kant’s theory suggests to me the idea that not only mathematics has something to do with the sublime but vice versa as well: the sublime always has something to do with mathematics. Let me unpack this idea a little.

Arguably the most famous passage from Kant is the first paragraph of the conclusion to his *Kritik der praktischen Vernunft* (1788) on “the starry heavens above me and the moral law within me”.⁵¹ Please, note that that passage is about *two infinities*: the infinity of the natural world and the infinity of the human personality. What is more, it is about the sublime: it is no coincidence that in the next paragraph, Kant begins with words on “the sublimity (Erhabenheit)” of the object of his inquiry. Surely, Kant’s famous words bring to mind Blaise Pascal’s

⁴⁹ I. Kant, *Critique of the Power of Judgment*, p. 136.

⁵⁰ Words by Jacob Rogozinski. *Ibid.*, p. 149. Italics mine – V.S.

⁵¹ I. Kant, *Critique of Practical Reason*, in: I. Kant, *Practical Philosophy*, transl. and ed. M.J. Gregor (The Cambridge Edition of the Works of Immanuel Kant), New York, NY 1996, pp. 269–270.

thoughts on infinity: the principle of double infinity and Pascal’s terror (effroi) (fragments S229/L198-S234/L202⁵²). In Pascal, the connection of the infinity in nature with mathematics (as well as its sublimity) is quite apparent.⁵³

In his *Kritik der Urteilskraft* (1790), Kant introduces two types of the sublime: “the mathematically sublime (das Mathematisch-Erhabene)” and “the dynamically sublime (das Dynamisch-Erhabene).”⁵⁴ The first presupposes a very great, hardly imaginable *magnitude* while the second one a very great, totally irresistible *power*. Though only the first type is called “mathematisch”, both of them are connected with mathematics, for Kant introduces not only *extensive* magnitudes, but *intensive* magnitudes as well, and it is the intensive magnitude of power that matters when we recognise any power as sublime. The standard (or first) application of mathematics is regulated by the principle of the axioms of intuition: “All intuitions are extensive magnitudes.”⁵⁵ That means that intuitions in question presuppose “successive synthesis (form part to part)”⁵⁶ in space and time. The so-called “second application of mathematics”⁵⁷ is regulated by the principle of the anticipations of perception: “In all appearances the real, which is an object of the sensation, has intensive magnitude, i.e., a degree.”⁵⁸ Gravity or weight, colour, heat, light, energy or power, etc., they all are intensive magnitudes, according to Kant. Intensive magnitude means here “a degree of influence on sense.”⁵⁹ In that case, apprehension “is not successive but instantaneous”, i.e., it never goes from parts to the whole but “fills only an instant”.⁶⁰ Nevertheless, those instants (of the same kind, of course) constitute a continuous scale of degrees, that makes them countable and, in general, mathematizable.

⁵² B. Pascal, *Pensées*, ed. and transl. R. Ariew, Indianapolis, IN 2004, pp. 57–64.

⁵³ Cf. T. Pavlovits, *Admiration, Fear, and Infinity in Pascal’s Thinking*, in: *Philosophy Begins in Wonder: An Introduction to Early Modern Philosophy, Theology and Science*, eds. M.F. Deckard, P. Losonczi, Cambridge 2011, pp. 119–126.

⁵⁴ I. Kant, *Critique of the Power of Judgment*, p. 131.

⁵⁵ I. Kant, *Critique of Pure Reason*, transl. and ed. P. Guyer, A.W. Wood (The Cambridge Edition of the Works of Immanuel Kant), New York, NY 1998, p. 286.

⁵⁶ *Ibid.*, p. 288.

⁵⁷ I. Kant, *Prolegomena*, transl. G. Hatfield, in: I. Kant, *Theoretical Philosophy after 1781*, eds. H. Allison, P. Heath (The Cambridge Edition of the Works of Immanuel Kant), New York, NY 2002, p. 100.

⁵⁸ I. Kant, *Critique of Pure Reason*, p. 290.

⁵⁹ *Ibid.*

⁶⁰ *Ibid.*, p. 291.

From empirical examples to divine symbols

In general, Kant is inclined to emphasise the conflict and to contrast human cognitive abilities, to separate theoretical and practical reason. Nevertheless, his aesthetics of the sublime provides a golden opportunity to make overt an interesting interplay of abilities in question, of both spheres of human reason and, hence, to bridge mathematics and theology.

The intimate interaction of the understanding (*διάνοια*) and imagination (*φαντασία*) in geometry was well known already in antiquity as Proclus attests.⁶¹ Kant continued that tradition when he asserted *constructivity* as a distinctive feature of mathematical cognition (A713/B741).⁶² Moreover, he extended this approach from geometry to arithmetic and algebra (and thus to the whole of mathematics) by introducing the idea of “symbolic construction” along with “ostensive or geometrical construction” (A717/B745).⁶³ In fact, both types of construction successfully combine in all parts of mathematics.

Two fundamental types of mathematical construction find their parallel in Kant’s duality of *schematism* and *symbolism*.⁶⁴ The reality of our concepts can be demonstrated, according to Kant, only through obtaining appropriate intuitions. The ideas of reason have no corresponding schemata in the theoretical sphere but can have symbolic representations in the practical sphere. For instance, “all of our cognition of God is merely symbolic”.⁶⁵

CONCEPTS	INTUITIONS
empirical concepts	examples
pure concepts of the understanding	schemata
ideas of reason	symbols

The idea of symbolic construction in mathematics shows that Kant recognises symbolism, along with schematism, not only in the practical sphere but in the theoretical one as well. All the more so, there is no real reason why one could not

⁶¹ See Proclus on a standard structure of any geometrical theorem or problem. Proclus, *Op. cit.*, pp. 9, 41, 157–164.

⁶² I. Kant, *Critique of Pure Reason*, p. 630.

⁶³ *Ibid.*, p. 632.

⁶⁴ I. Kant, *Critique of the Power of Judgment*, p. 225–227.

⁶⁵ *Ibid.*, p. 227.

use mathematics symbolically in the practical sphere, even if it would not be “the actual schema for the concept but only a symbol for reflection”.⁶⁶

The infinite in the finite

In a way, the notion of *symbol* provides an opportunity to synthesise the three Kant’s conceptual layers and, hence, the three types of the infinite and, hence, the finite and the infinite. Kant’s successors were happy to use the opportunity and to interpret the sublime through such a synthesis. According to Schelling, *sublimity* is constituted by “the informing of the infinite into the finite”. He continues: “wherever we encounter the infinite being taken up into the finite as such—whenever we distinguish the infinite within the finite—we judge that the object in which this takes place is sublime” (*The Philosophy of Art*, § 65).⁶⁷ In this case, the finite turns into “a symbol of the infinite”.⁶⁸ “The finite” in the case of mathematics is its diagrams and strings of characters. According to Hegel, “when the symbol is developed independently in its own proper form, it has in general the character of sublimity”.⁶⁹ “Symbolic art”, or “the art of sublimity”, he characterises as “the sacred art as such”.⁷⁰ They called such a synthesis of the finite and the infinite “dialectics”, but perhaps “*mystical vision*” is a better name. For instance, a US science journalist John Horgan gives the following as a formulation of mysticism: “awestruck perception of the infinite in the finite”.⁷¹ William Blake expressed this very idea in famous verses: “To see a World in a Grain of Sand, / And a Heaven in a Wild Flower, / Hold Infinity in the palm of your hand, / And Eternity in an hour”.⁷²

Mathematics does just this: *makes the infinite in the finite visible*. Moreover, mathematics boasts a special affinity with the infinite. While attempts to define

⁶⁶ *Ibid.*, p. 226.

⁶⁷ F.W.J. Schelling, *The Philosophy of Art*, Minneapolis, MN 1989, pp. 85–86.

⁶⁸ *Ibid.*, pp. 62–69, 79, 87–90.

⁶⁹ G.W.F. Hegel, *Aesthetics: Lectures on Fine Art*, New York, NY 1975, Vol. I, p. 303.

⁷⁰ *Ibid.*, pp. 372–373.

⁷¹ J. Horgan, *Rational Mysticism: Dispatches from the Border between Science and Spirituality*, New York, NY 2003, p. 215.

⁷² W. Blake, *Auguries of Innocence*, the Pickering Manuscript, c.1801–1803, in: *The Poetical Works: A New and Verbatim Text from the Manuscript Engraved and Letterpress Originals*, ed. J. Sampson, Oxford 1947, p. 288.

mathematics as *the science of the infinite* is a modern story (the 1920s, H. Weyl and E. Zermelo),⁷³ the very idea of the unique relationship between infinity and mathematics dates back to Aristotle, as was already mentioned above. In *Physics* (III, 4–8), he reduces different variants of the usage of the word “infinity” to the only proper one: infinity as an accident of quantity (that is of magnitudes and numbers as well as time, place, and movement). According to Aristotle, infinity has only a *potential* existence: “For generally the infinite has this mode of existence: one thing is always being taken after another, and each thing that is taken is always finite, but always different” (III, 6, 206a).⁷⁴ It means, above all, that mathematicians work with the infinite *through the finite*.⁷⁵

The stars of the heaven and the sand of the sea

Let us consider a biblical example. God promises Abraham: “I will greatly multiply your descendants so that they will be as countless as the stars in the sky or the grains of sand on the seashore” (*Genesis* 22:17; cf. *Genesis* 32:12; *Hosea* 1:10; *Jeremiah* 33:22). The stars of the heaven and the sand of the sea were seen as something truly sublime; they filled people with awe for they were too numerous to be counted one by one in any finite time within human life, which is too short. It is possible only for immortal God, not for a mortal man. Something extremely big was almost equated with the infinite. To be able to handle the infinite (or very big) well means to obtain divine powers. This idea was made perfectly explicit in the apocryphal *Greek Apocalypse of Ezra* (2:32–3:3):

(And God said,) ‘Count the stars and the sand of the sea; and if you will be able to count this, you will also be able to argue the case with me.’ And the prophet

⁷³ Cf. H. Weyl, *Die heutige Erkenntnislage in der Mathematik* (1925), in: *Gesammelte Abhandlungen*, Berlin 1968, Bd. II, p. 511; E. Zermelo, *Thesen über das Unendliche in der Mathematik / Theses Concerning the Infinite in Mathematics* (s1921), in: E. Zermelo, *Collected Works / Gesammelte Werke*, Berlin 2010, Vol. I, pp. 306–307; E. Zermelo, *Vortrags-Themata für Warschau 1929 / Lecture Topics for Warsaw 1929* (s1929b), in: *ibid.*, pp. 382–383.

⁷⁴ Aristotle, *Complete Works*, Vol. 1, p. 351.

⁷⁵ This is one of the central ideas in D. Hilbert’s *Über das Unendliche*, “*Mathematische Annalen*” 1926, Vol. 95, pp. 161–190. David Hilbert was lecturing at Göttingen on the same subject in the winter semester of 1924–25, trying to clarify the nature and meaning of the *infinite* in mathematics.

said, ‘Lord, you know that I bear human flesh. And how can I count the stars of the heaven and the sand of the sea?’⁷⁶

Nevertheless, mathematicians pretend to fulfil the job rejected by Ezra as Archimedes famously showed in his *Psammites* (The Sand Reckoner). They obtain a divine power over very big numbers (about 10^{63} grains of sand in the case of Archimedes) and even infinity. According to Scott J. Aaronson, a theoretical computer scientist at MIT, “one could define science as reason’s attempt to compensate for our inability to perceive big numbers”.⁷⁷ Nevertheless, the “big number phobia” is real and still with us. It is the other side of the fascination and witness to the sublimity of big numbers.

The sublime as the key to mathematical beauty

Is it true that the chiliagon gives us a sublime feeling while the triangle is not? Is it true that the triangle can be beautiful but never sublime? I do not think so. Kant is inclined to contrast *the sublime* to *the beautiful* through the opposition of the infinite and the finite, the formless and the form, quantity and quality, reason and the understanding.⁷⁸ Such oppositions seem rather superficial: we are forced to grasp complicated forms of the natural world with the help of basic regular geometrical patterns,⁷⁹ and the peculiar beauty of such patterns is intimately connected with the possibility of encapsulating the infinite in the finite.

Already, the Ancient Greek geometers were extremely sensitive to the subject. Consider such fundamental geometrical objects as *the straight line* and *the circle*. A straight line (which was, for the Ancient Greeks, a segment of a line, though extendable ad infinitum⁸⁰) was defined in Euclid as “a line which lies evenly with

⁷⁶ *Greek Apocalypse of Ezra (Second to Ninth Century AD), a New Translation and Introduction by M.E. Stone*, in: *The Old Testament Pseudepigrapha*, Vol. I: *Apocalyptic Literature and Testaments*, ed. J.H. Charlesworth, Peabody, MA 1983, p. 573.

⁷⁷ S.J. Aaronson, *Who Can Name the Bigger Number?* 1999. Retrieved from <http://www.scottaaronson.com/writings/bignumbers.pdf>

⁷⁸ I. Kant, *Critique of the Power of Judgment*, p. 128.

⁷⁹ Consider classic techniques in the art of drawing with their basic geometrical frame. Cf. *The Art of Basic Drawing*, Walter Foster Publishing, Laguna Hills, CA 2007. They uncover the geometrically informed constructive activity of the human mind famously emphasised by Kant (BXI–XII). I. Kant, *Critique of Pure Reason*, pp. 107–108.

⁸⁰ Euclid’s postulate 2.

the points on itself”.⁸¹ It means that to be straight, any segment of the line must be straight, so to say, in each of its points, which are infinitely many. Similarly, a circle was defined as having all the radii equal, which make it circular, once again, in infinitely many points. The three-dimensional analogue of the circle is the sphere. Pythagoreans singled them out from all shapes. We are told that Pythagoras “held that the most beautiful figure is the sphere among solids, and the circle among plane figures”.⁸² Proclus is more explicit:

The first and simplest and most perfect of the figures is the circle. It is superior to all solid figures because its being is of a simpler order, and it surpasses other plane figures by reason of its homogeneity and self-identity. [...] Hence, whether you analyze the cosmic or the supercosmic world, you will always find the circle in the class nearer the divine.⁸³

Even the beauty of the simplest geometrical figures, such as the circle and the straight line, is because of the covert presence of infinity in their inner structure, i.e., *the sublimity at the bottom of their beauty*. That presence is apparently the reason why *the circle* and *the sphere* have been immensely popular as the symbols for the divine reality. They were preferred to the straight line and the plane for the latter two (if taken in Greek interpretation) are always unfinished and incomplete.

Mathematical sublimity too often adds to mathematical beauty, and, perhaps, even more than that. I would like to make a conjecture that *mathematical sublimity keeps the secret of mathematical beauty in general*.⁸⁴

⁸¹ *The Thirteen Books of Euclid's Elements*, transl., intr. and commentary T. Heath, Cambridge 1908, Vol. 1, pp. 153, 165–169.

⁸² Diogenes Laertius, VIII, 35. See *Diogenes Laertius, Lives of Eminent Philosophers, with an English Translation by R.D. Hicks* (Loeb Classical Library), London; New York 1925, Vol. II, pp. 350–351.

⁸³ Proclus, *Op. cit.*, p. 117.

⁸⁴ Francis Hutcheson adhered to this opinion (though without using the word “sublime”) in his anonymously published 1725 treatise on aesthetics. This treatise contained a special section on “the Beauty of Theorems” that elaborated upon the subject. The secret of mathematical beauty, according to Hutcheson, lies in the fact that “in one Theorem we may find included, with the most exact Agreement, an infinite Multitude of particular Truths; nay, often an Infinity of Infinities”. For example, the Pythagorean theorem is comprised of truths about infinitely many particular right-angled triangles that vary in form and size. He also refers to differential calculus, in which one theorem embraces infinite species of curves, infinite sizes within each species, and infinite points within each curve. See F. Hutcheson, *An Inquiry into the Original of Our Ideas of*

Let us return to the question of whether the triangle can be perceived as sublime or not. The true triangle, according to Proclus, is the equilateral one; all others are *distortions* of the paradigm.⁸⁵ All regular triangles are similar; they all have the same form. On the contrary, there are *infinitely* many irregular triangles that lack similarity. If the trisector theorem, discovered by Frank Morley in 1899,⁸⁶ had been known in the time of Proclus, he would have possibly commented on it in the following manner: in the very heart of any triangle, no matter how much distorted, one can always find its true paradigm unchanged, an equilateral triangle in all its perfection and splendour. “It is one of the most astonishing and totally unexpected theorems in mathematics and, jewel that it is, for sheer beauty it has few rivals”.⁸⁷ Such is the verdict of some 20th-century mathematicians. What can be said about the sublimity of Morley’s theorem? Its beauty owes a lot not only to its simplicity, which stayed undiscovered for such a long time, but to the unexpectedness of the result itself.⁸⁸ All the *infinite* variety of triangles turned out to be unified and, hence, grasped in a new way, through the regular form of their inner triangle, formed by the trisectors. Here, we also have a sort of encapsulating of the infinite in the finite, which is supposed to provoke a sublime feeling.

In search of The Book Proof

Mathematics cannot be reduced to the contemplation of mathematical objects and formulation of propositions or theorems, grasping some of their properties, in an oracle-like manner; first and foremost, it is about finding mathemati-

Beauty and Virtue in Two Treatises, ed. W. Leidhold, Indianapolis 2004, pp. 36–37. Please, note that my point is slightly different from that of Hutcheson and may be considered as complementary to his: while he primarily talks about mathematical propositions, I talk about mathematical objects.

⁸⁵ *Ibid.*, p. 133.

⁸⁶ Here is its formulation: “The three intersections of the trisectors of the angles of a triangle, lying near the three sides respectively, form an equilateral triangle”. See C.O. Oakley, J.C. Baker, *The Morley Trisector Theorem*, “American Mathematical Monthly” 1978, Vol. 85, No. 9, p. 738.

⁸⁷ *Ibid.*

⁸⁸ Gian-Carlo Rota claims that Morley’s theorem “is unquestionably surprising, but neither the statement nor any of the proofs of the theorem can be viewed as beautiful”. G.-C. Rota, *The Phenomenology of Mathematical Beauty*, “Synthese” 1997, Vol. 111, No. 2, p. 172. Beauty is in the eye of the beholder even in the case of mathematics.

cal proofs. What about the sublimity of the proofs? For instance, Ian Hacking speaks of the “awe” that mathematical proofs inspire and says that he shares in that awe.⁸⁹ It is well known that mathematicians never get tired of looking for new proof for an old result.⁹⁰ They also discuss the beauty and perfection of the proofs. When Bertrand Russell wrote about “the sense of being more than man”⁹¹ while doing mathematics, he was thinking not of the objects or the results but the mathematical proofs, especially of the intrinsic perfection of their inner logic. He also took an intense interest in the global structure of mathematics as a system of interconnected theories, and compared mathematics as a whole with a temple or a palace⁹², repeatedly using architectural metaphors. He wrote that “the rules of logic are to mathematics what those of structure are to architecture”.⁹³ Russell provides a truly good example of a mathematician of a “cathedral-builder” type. A great Hungarian mathematician Paul Erdős gives another impressive example.

One of Erdős’s results—an elegant eight-line proof, which is famous for introducing probabilistic arguments in combinatorics—recently evoked the following remark from an American mathematician Michael H. Harris:

It’s a proof many mathematicians would be tempted to call beautiful. But can such a proof be sublime in the sense of Kant and Edmund Burke? Can meditating about finite sets inspire awe and terror? Or are these sentiments reserved for the contemplation of infinity?⁹⁴

“Dipping into the mathematical papers of Paul Erdős is like wandering into Aladdin’s Cave. The beauty, the variety and the sheer wealth of all that one finds are quite overwhelming”.⁹⁵ With such words, Béla Bollobás, one of the mathematicians of Erdős’s circle, begins an overview of his life and work. The words “beauty” and “beautiful” constantly appear when mathematicians discuss Erdős’s

⁸⁹ I. Hacking, *Husserl on the Origins of Geometry*, in: *Science and the Life-World: Essays on Husserl’s ‘Crisis of European Sciences’*, eds. D. Hyder, H.-J. Rheinberger, Stanford 2010, p. 65.

⁹⁰ J.W. Dawson Jr, *Why Do Mathematicians Re-prove Theorems?*, “*Philosophia Mathematica*” 2006, Series III, Vol. 14, No. 3, pp. 269–286.

⁹¹ See the epigraph from Russell at the very beginning of this paper.

⁹² B. Russell, *The Study of Mathematics*, in: *Mysticism and Logic and Other Essays*, pp. 58–59, 67–68.

⁹³ *Ibid.*, p. 61.

⁹⁴ M. Harris, *Mathematics Without Apologies: Portrait of a Problematic Vocation*, Princeton 2015, pp. 192–193.

⁹⁵ B. Bollobás, *Paul Erdős: Life and Work*, in: *The Mathematics of Paul Erdős I*, eds. R.L. Graham, J. Nešetřil, S. Butler, 2nd ed., New York, NY 2013, p. 1.

mathematics. But it seems to be more to it than that. In a striking resemblance to Russell (despite all the differences), Erdős, being an agnostic,⁹⁶ was fond of using religious vocabulary while speaking of mathematics. Lecturing on mathematics was famously called “preaching”. As Bollobás puts it, in his language “a mathematician could *preach*, usually to the converted”.⁹⁷ One of Erdős’s much younger friends and co-author, Joel Spencer, tried to put an eluding idea of his elder friend and mentor’s quasi-religious attitude towards mathematics into words:

What drew so many of us into his circle? What explains the joy we have in speaking of this gentle man? Why do we love to tell Erdős stories? I’ve thought a great deal about this and I think it comes down to a matter of belief, or faith. We know the beauties of mathematics and we hold a belief in its transcendent quality. [...] Mathematical truth is immutable, it lies outside physical reality. [...] This is our belief, this is our core motivating force. Yet our attempts to describe this belief to our nonmathematical friends is akin to describing the Almighty to an atheist. Paul embodied this belief in mathematical truth. His enormous talents and energies were given entirely to the Temple of Mathematics. He harbored no doubts about the importance, the absoluteness, of his quest. To see his faith was to be given faith. The religious world has a name for such people—they are called saints.⁹⁸

Please, note how persistent Spencer is in using religious vocabulary while talking about Erdős the mathematician! He continues:

I do hope that one cornerstone of Paul’s, if you will, theology will long survive. I refer to The Book. The Book consists of all the theorems of mathematics. For each theorem there is in The Book just one proof. It is the most aesthetic proof, the most insightful proof, what Paul called The Book Proof. When one of Paul’s myriad conjectures was resolved in an “ugly” way Paul would be very happy in congratulating the prover but would add, “Now, let’s look for The Book Proof.” This platonic ideal spoke strongly to those of us in his circle. The mathematics was there, we had only to discover it. [...] In the summer of

⁹⁶ Cf. P. Hoffman, *The Man Who Loved Only Numbers: The Story of Paul Erdős and the Search for Mathematical Truth*, London 1998, p. 26.

⁹⁷ B. Bollobás, *To Prove and Conjecture: Paul Erdős and His Mathematics*, “The American Mathematical Monthly” 1998, Vol. 105, No. 3, p. 212.

⁹⁸ J. Spencer, *Erdős Magic*, in: *The Mathematics of Paul Erdős I*, eds. R.L. Graham, J. Nešetřil, S. Butler, 2nd ed., New York, NY 2013, p. 45.

1985 I drove Paul to Yellow Pig Camp—a mathematics camp for talented high school students. [...] In my introduction to his lecture I discussed The Book but I made the mistake of describing it as being “held by God.” Paul began his lecture with a gentle correction that I shall never forget. “You don’t have to believe in God,” he said, “but you should believe in The Book.”⁹⁹

Not everyone in Erdős’s circle would agree with such an interpretation. For example, Bollobás is far less enthusiastic on the subject:

Occasionally Erdős talked of *The Book*: a transfinite book whose pages contain all the theorems and their best possible proofs. Unfortunately, went on Erdős, The Book is held by God who, being malicious towards us, only very rarely allows us to catch a glimpse of a page. But when that happens, then we see mathematics in all its beauty. In writing and talking about Erdős, The Book is frequently overemphasized: he himself always insisted that this is only a joke, which should not be taken seriously, lest it damages the mathematics we like. By its very nature, a book proof tends to be short and snappy: it is as if lightning allowed us to see some detail clearly. When it comes to substantial results, the best we can feel is that the *global idea* of the proof is from The Book.¹⁰⁰

Nevertheless, both Spencer and Bollobás argue to the same end. The use of religious terminology by Erdős cannot be taken at face value, he was surely joking; but a good joke usually both covers and uncovers something quite serious, and his complete devotion to mathematics was extremely serious. His biographer Paul Hoffman calls him for that reason “a mathematical monk”.¹⁰¹ One of Erdős’s favourite sayings runs: “Every human activity, good or bad, *except mathematics*, must come to an end”.¹⁰² A person who stopped doing mathematics was, in his language, “dead”.¹⁰³ It was not just about beauties, as he proved by his own life. Erdős obviously “had already run beyond beauty” (to quote Plotinus once again) and had got at least into the sphere of sublime if not numinous.

⁹⁹ *Ibid.*, pp. 45–46.

¹⁰⁰ B. Bollobás, *To Prove and Conjecture*, pp. 219–220.

¹⁰¹ P. Hoffman, *Op. cit.*, p. 25.

¹⁰² B. Bollobás, *To Prove and Conjecture*, p. 209.

¹⁰³ P. Hoffman, *Op. cit.*, pp. 8, 133.

What makes a mathematical proof provoke the sublime feeling? In my opinion, it is an inner “tension” or “τόνος” between the two poles, the “finite” and the “infinite” ones, of the dipolar structure of any symbol that is quite recognisable in mathematical proofs. The finite pole is the so-called “simplicity” of a proof, which can mean a lot of different things: clarity, lucidity, and perspicuity, as well as conciseness and compactness, easy surveyability or minimalism of conceptual prerequisites. The infinite pole is formed by *generality* or considerable coverage and *unexpectedness* of the new relations the proof uncovers. The more intense is the interplay between the poles, the more sublime and, hence, beautiful is the proof. That is what “The Book Proof” means.

What is The Book, anyway? What does that mathematical “paradise” look like? Does it look like a chaotic collection of separate proofs¹⁰⁴ or a unified theory, embracing the whole of mathematics? Note that Bertrand Russell was the one to do his best at creating a version of The Book. I have in mind his unfinished project of *Principia Mathematica*,¹⁰⁵ whose foci of interest were ultimate logical coherence and unification. The similar architectonic vision seemed to guide later the Bourbaki project,¹⁰⁶ though they were not so scrupulous about logic and the finality of a mathematical edifice. Both unification projects just mentioned have failed, and mathematicians have to reconcile themselves to having just a collection of more or less extensive theoretical fragments.

Nevertheless, some of the mathematicians foresee more than mere fragments in the mathematical theories we have. They are still dreaming of a final, unified and comprehensive theory, i.e., the majestic “cathedral” of pure mathematics, whose might and splendour are visible only to the initiated, thus demonstrating a theological attitude towards doing mathematics. Both the believers and the agnostics, they are highly sensitive to the sublimity of mathematics, and they do their work in great hope that they are not labouring in vain.

¹⁰⁴ Cf. M. Aigner, G.M. Ziegler, *Proofs from THE BOOK*, 6th ed., Berlin 2018. It was first published in 1998. Erdős, who died in the summer of 1996, had approved of the idea of such a publication and gave his suggestions on the content. According to the authors, “to a large extent this book reflects the views of Paul Erdős as to what should be considered a proof from The Book” (p. V). As would be expected, it is no more than a collection of beautiful fragments, and it often gives several proofs for the same result.

¹⁰⁵ A.N. Whitehead, B. Russell, *Principia Mathematica*, Vol. I–III, Cambridge 1910–1913.

¹⁰⁶ N. Bourbaki, *The Architecture of Mathematics*, “The American Mathematical Monthly” 1950, Vol. 57, No. 4, pp. 221–232.

Concluding remarks

In this paper, I have ventured several hypotheses concerning the nature of mathematical sublime. I just hypothesise and illustrate my point of view, giving a series of examples. My point is to introduce new topics for further consideration, not to provide full arguments or ultimate proofs. Here is a list of my main conjectures:

- (1) The *aesthetic* dimension of mathematics is intimately linked with the *religious* one. The aesthetic feelings of beauty, and especially sublime, often serve as a secular substitute for religious awe or the numinous feeling. Such a substitution works not only in art but in mathematics as well. To talk about it, I propose the terms “numinous mathematics” and “mathematical sublime”.
- (2) To account for the connection between aesthetic and religious dimensions of mathematics, I suggest revitalising a classical understanding of mathematics as *the science of the infinite*. More precisely, mathematics can be seen as the science of forming the infinite into the finite.
- (3) Mathematical sublimity keeps the secret of mathematical beauty in general. In my opinion, the beautiful in mathematics primarily presupposes a considerable tension between the simplicity and clarity of *finite form* and unexpected richness of the *infinite content* grasped by this form, that is, presupposes the sublime.

All the issues listed above, surely need further research to be elaborated and proved or disproved.

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Summary

The paper deals with aesthetic and religious dimensions of mathematics. These dimensions are considered as closely connected, though reciprocally non-reducible. "Mathematical beauty" is already firmly established as a term in the philosophy of mathematics. Here, an attempt is made to bring forward two additional candidates: "mathematical sublime" and "numinous mathematics". The last one is meant to designate the recognition of some mathematical practices as inspiring anticipation of the meeting with the divine reality or producing a feeling of its presence. The first one is used here to designate the related feelings in disguise, i.e., being reinterpreted or transferred from the straightforwardly religious to the aesthetic sphere. Taking Kant's theory of the sublime as a starting point, the paper introduces a related account of it that treats mathematical beauty through mathematical sublimity as a more fundamental category. Within this account, religious experience, the aesthetics of the sublime and mathematical practice are closely interlinked through an appropriate interpretation of the idea of the infinite. Both mathematical and art symbolism are seen as an endeavour to represent the infinite within the finite, which correlates well with the definition of mathematics as "the science of the infinite" (Hermann Weyl).

Key words: philosophy of mathematics, philosophy of mathematical practice, mathematical aims, mathematical beauty, the sublime, numinous, the infinite, symbolism

Streszczenie

„Budowniczy katedr”: matematyka a wzniosłość

Artykuł poświęcony jest estetycznemu wymiarowi matematyki, a także jego wymiarowi religijnemu. Wymiary te rozważane są jako silnie ze sobą powiązane, choć nie są do siebie sprowadzalne. „Piękno matematyczne” ugruntowało się już jako termin w filozofii matematyki. Podjęto tu próbę wysunięcia dodatkowych kandydatów: „matematyczna wzniosłość” i „matematyka numinotyczna”. Drugi z nich odnosi się do uznania pewnych praktyk matematycznych jako inspirujących do antycypacji spotkania z boską rzeczywistością lub jako wywołujących poczucie jej obecności. Z kolei pierwszy – do związanych z tym odczuć w „przebraniu”, to jest zreinterpretowanych i przeniesionych ze sfery wprost religijnej do estetycznej. Wychodząc od teorii wzniosłości Kanta, artykuł proponuje ujęcie matematycznego piękna poprzez matematyczną wzniosłość jako kategorię podstawową. W tym zakresie doświadczenie religijne, estetyka wzniosłości i praktyka matematyczna są wzajemnie silnie powiązane poprzez odpowiednią interpretację idei nieskończoności. Zarówno symbolizm matematyczny, jak i symbolizm w sztuce są tu postrzegane jako próba przedstawienia nieskończoności w tym, co skończone, co dobrze koreluje z definicją matematyki jako „nauki o nieskończoności” (Hermann Weyl).

Słowa kluczowe: filozofia matematyki, filozofia praktyki matematycznej, cele matematyczne, matematyczne piękno, wzniosłość, numinosum, nieskończoność, symbolizm